Off-shell Bethe ansatz for the XXZ vertex model and solution of the trigonometric Knizhnik-Zamolodchikov equations

1994 J. Phys. A: Math. Gen. 27 7753

(http://iopscience.iop.org/0305-4470/27/23/018)

J. Phys. A: Math. Gen. 27 (1994) 7753–7761. Printed in the UK

Off-shell Bethe ansatz for the *XXZ* vertex model and solution of the trigonometric Knizhnik–Zamolodchikov equations

H M Babujian†‡
Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany

Received 1 August 1994

Abstract. We prove that the wavevector of the off-shell Bethe ansatz equation for the inhomogeneous *XXZ* vertex model in the quasiclassical limit gives the solution of the trigonometric Knizhnik–Zamolodchikov equation. We also observe that this solution is the quasiclassical limit of the solution of the quantum Knizhnik–Zamolodchikov equation corresponding to $U_q(\widehat{sl}_2)$.

1. Introduction

Conformal field theory describes the critical behaviour of the two-dimensional physical systems, many of which are exactly integrable. In this case we have found the structure connected to the Yang–Baxter equation, which guarantees exact integrability. Usually the exact integrable homogeneous vertex model and its connection with the conformal field theory [4] or quantum field theory [5] has been considered. The next step in understanding this connection is the investigation of the inhomogeneous vertex models. Here we consider the inhomogeneous vertex model, where to each vertex we associate two parameters: the global spectral parameter λ and the disorder parameter z. The vertex weight matrix $\mathfrak{R}$ depends on $\lambda - z$. Hence the transfer matrix of the vertex model now depends on the disorder parameters $z_i, i = 1, \ldots, N$. Due to the additivity of the spectral parameter, transfer matrices with different values of spectral parameters commute which each other [1], which means that the model is integrable. If we have some rational solution of the Yang–Baxter equation $\mathfrak{R}(\lambda; \eta)$ and the transfer matrix $T(\lambda|\{z\})$, then by construction of the algebraic Bethe ansatz [6] we have an equation

$$T(\lambda|\{z\})\Phi(\lambda_1, \ldots, \lambda_n|\{z\}) = \Lambda(\lambda, \lambda_1, \ldots, \lambda_n|z)\Phi(\lambda_1, \ldots, \lambda_n|\{z\}) - \sum_{\alpha=1}^{m} \frac{F_\alpha \Phi_\alpha}{\lambda - \lambda_\alpha} \tag{1}$$

where

$$\Phi(\lambda_1, \ldots, \lambda_n|\{z\}) = \Phi(\lambda_1, \ldots, \lambda_m|z_1, \ldots, z_N)$$

is a Bethe wavevector and

$$\Phi_\alpha = \Phi(\lambda_1, \ldots, \lambda_{\alpha-1}, \lambda, \lambda_{\alpha+1}, \ldots, \lambda_m|z_1, \ldots, z_N).$$

† Permanent address: Yerevan Physics Institute, Alichanian Brothers 2, Yerevan, 375036 Armenia.
‡ Partially supported by the grant 211-5291 YPI of the German Bundesministerium für Forschung and Technolgie, the International Soros Foundation and through the VW project 'Cooperation with scientists from CIS'.

Furthermore, $F_\alpha(\lambda_1,\ldots,\lambda_m|z_1,\ldots,z_N)$ and $\Lambda(\lambda,\lambda_1,\ldots,\lambda_m|z_1,\ldots,z_N)$ are some functions and η is a Planck-type parameter, such that $\mathfrak{R}(\lambda;0)=I\otimes I$; this means that $\mathfrak{R}$ is a unit operator at the $\eta=0$ point. In the Bethe ansatz we impose the condition $F_\alpha=0$ and after this we have the exact eigenvalue problem such that the vector Φ becomes an eigenvector and Λ an eigenvalue of $T(\lambda|z)$, respectively. In fact the condition $F_\alpha=0$ is the mass-shell condition. If we keep all the 'unwanted' terms in (1), then the vector Φ in general satisfies the equation (1). We call equation (1) the off-shell Bethe ansatz equation (OSBAE). In [7–9] we have constructed the solution of the Knizhnik–Zamolodchikov equation, which is the differential equation for the N-point correlation function $\Psi(z_1,\ldots,z_N)$ in the WZNW theory [10]

$$\kappa\frac{\partial\Psi}{\partial z_k}=\sum_{i\neq k}^{N}\frac{t_k^a\otimes t_i^a}{z_k-z_i}\Psi \tag{2}$$

using the OSBAE (1) in the quasiclassical limit, when $\eta\to 0$. In this limit we have the off-shell Bethe ansatz problem for the Gaudin theory of non-local magnets [2, 3] $H_k, k=1,\ldots,N$:

$$H_k=\sum_{i\neq k}^{N}\frac{t_k^a\otimes t_i^a}{z_k-z_i}. \tag{3}$$

In (2) and (3) the t_i^a, $a=1,\ldots,\dim(\mathfrak{G})$, represent the generators of the simple Lie algebra $\mathfrak{G}$ and act non-trivialy in the representation spaces $V^i, i=1,\ldots,N$. The vector-valued function $\Psi(z_1,\ldots,z_N)$ is the correlation function of the primary fields in the WZNW theory and $\Psi(z_1,\ldots,z_N)\in V^1\otimes\ldots\otimes V^N$. The solution of (2) as in [11] is

$$\Psi(z_1,\ldots,z_N)=\oint\ldots\oint\chi(\lambda|z)\varphi(\lambda|z)\mathrm{d}\lambda_1,\ldots,\mathrm{d}\lambda_m. \tag{4}$$

Here $\varphi(\lambda|z)=\varphi(\lambda_1,\ldots,\lambda_m|z_1,\ldots,z_N)$ is the quasiclassical limit of the Bethe wavevector $\Phi(\lambda_1,\ldots,\lambda_m|z_1,\ldots,z_N)$ in the sense that

$$\Phi(\lambda_1,\ldots,\lambda_m|z_1,\ldots,z_N)\to(2\eta)^m\varphi(\lambda_1,\ldots,\lambda_m|z_1,\ldots,z_N) \tag{5}$$

and in fact it is the Bethe wavevector for Gaudin magnets (3), but off mass shell. The scalar function $\chi(\lambda|z)=\chi(\lambda_1,\ldots,\lambda_m|z_1,\ldots,z_N)$ is constructed from the quasiclassical limit of the $\Lambda(\lambda=z_k,\lambda_1,\ldots,\lambda_m|z_1,\ldots,z_N), k=1,\ldots,N$ and $F_\alpha(\lambda_1,\ldots,\lambda_m|z_1,\ldots,z_N)$. This representation of the N-point correlation function in WZNW theory shows a deep connection between the inhomogeneous vertex models and the WZNW theory. The understanding of this fine structure seems to provide us with new knowledge of exact integrability and conformal field theory in two dimensions. The application of this method in the case of trigonometric and elliptic solutions of the Yang–Baxter equation is also interesting in connection with the quantum Knizhnik–Zamolodchikov equation [12–14]. In this paper we consider the trigonometric solution of the Yang–Baxter equation and the corresponding trigonometric Knizhnik–Zamolodchikov equation [15]. More precisely, we replace in equation (2) the right-hand side, which corresponds to the rational Gaudin magnet (3), by the analogous expression of the trigonometric Gaudin [2, 3] magnet (38). Here we construct the off-shell Bethe ansatz equation for the *XXZ* inhomogeneous vertex model (section 2) and in the quasiclassical limit (section 3) we construct the solution of the trigonometric Knizhnik–Zamolodchikov equation (section 4). In section 5 we observe that the solution thus obtained is the quasiclassical limit of the solution of the quantum Knizhnik–Zamolodchikov equation corresponding to $U_q(\widehat{sl}_2)$.

2. Inhomogeneous *XXZ* vertex model

The solution of the inhomogeneous *XXZ* vertex model is completely analogous to the homogeneous case. We briefly describe this use of the results in [16, 17]. We use three types of $\mathfrak{R}$-matrices: $\mathfrak{R}^{12}_{\sigma}, \mathfrak{R}^{12}_{\sigma s}$ and $\mathfrak{R}^{12}_{s}$ acting in $V_\sigma \otimes V_\sigma$, $V_\sigma \otimes V_s$ and $V_s \otimes V_s$. These $\mathfrak{R}$-matrices satisfy the following three types of Yang–Baxter equations:

$$\mathfrak{R}^{12}_{\sigma s}(\lambda-\mu)\mathfrak{R}^{13}_{\sigma}(\lambda)\mathfrak{R}^{23}_{\sigma s}(\mu) = \mathfrak{R}^{23}_{\sigma s}(\mu)\mathfrak{R}^{13}_{\sigma}(\lambda)\mathfrak{R}^{12}_{\sigma s} \tag{6}$$

$$\mathfrak{R}^{12}_{\sigma s}(\lambda-\mu)\mathfrak{R}^{13}_{s}(\lambda)\mathfrak{R}^{23}_{s} = \mathfrak{R}^{23}_{s}(\mu)\mathfrak{R}^{13}_{s}(\lambda)\mathfrak{R}^{12}_{\sigma s}(\lambda-\mu) \tag{7}$$

$$\mathfrak{R}^{12}_{s}(\lambda)\mathfrak{R}^{13}_{s}(\lambda+\mu)\mathfrak{R}^{23}_{s}(\mu) = \mathfrak{R}^{23}_{s}(\mu)\mathfrak{R}^{13}_{s}(\lambda+\mu)\mathfrak{R}^{12}_{s}(\lambda). \tag{8}$$

In (6), $\mathfrak{R}^{12}_{\sigma}(\lambda)$ is a solution with $\dim V_\sigma = 2$, which is the well known *XXZ* spin-half $\mathfrak{R}$-matrix [3]

$$\mathfrak{R}^{12}_{\sigma}(\lambda) = \tfrac{1}{2}W_0(\lambda)I^1 \otimes I^2 + \tfrac{1}{2}\sum_{i=1}^{3} W_i(\lambda)\sigma^1_i \otimes \sigma^2_i \tag{9}$$

$$\begin{aligned} W_0(\lambda) &= 2\frac{\cos(\eta)\sin(\lambda+\eta)}{\sin(2\eta)} \\ W_3 &= 2\frac{\sin(\eta)\cos(\lambda+\eta)}{\sin(2\eta)} \\ W_1 &= W_2 = 1. \end{aligned} \tag{10}$$

Here I^1, I^2 are unit operators and $\sigma_i = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices acting on the first and second subspaces, respectively. The matrix $\mathfrak{R}^{12}_{\sigma s}$ is given as

$$\mathfrak{R}^{12}_{\sigma s}(\lambda) = \sum_{i=0}^{3} W_i(\lambda)\sigma^1_i \otimes I^2_i. \tag{11}$$

In (11) $\sigma_0 = I$ is the unit operator and the operators I_i are given as follows:

$$\begin{aligned} I_0 &= \frac{\cos(2\eta S^3)}{2\cos(\eta)} \qquad I_3 = \frac{\sin(2\eta S^3)}{2\sin(\eta)} \\ I_+ &= I_1 + \mathrm{i}I_2 = S^+ f(S^3) \qquad I_- = I_1 + \mathrm{i}I_2 = f(S^3)S^- \\ f(S^3) &= \frac{1}{\sin(2\eta)}\sqrt{\frac{\sin(2\eta(S^3+s+1))\sin(2\eta(s-S^3))}{(S^3+s+1)(s-S^3)}}. \end{aligned}$$

In fact I_3, I_+ and I_- are the generators of the $sl_q(2)$ algebra and together with I_0 they form the generators of the Sklyanin algebra [18]. In (11), $\mathfrak{R}^{12}_{\sigma s}(\lambda)$ acts on the space $V_\sigma \otimes V_s$ and $S_a = (S_1, S_2, S_3)$ are the matrices of the SU(2) algebra in arbitrary dimension, which means that $\dim V_s = 2s+1$ and $S_a S_a = s(2s+1)$. It is possible to find $\mathfrak{R}^{12}_{s}(\lambda)$ from (7) by a recurrence relation. This $\mathfrak{R}$-matrix was found in explicit form in [19]. Now let us introduce the disorder parameters $z_1, \ldots, z_N$ and as usual consider the monodromy operator [6]

$$J(\lambda|z) = \mathfrak{R}^{0N}_{\sigma s_N}(\lambda - z_N), \ldots, \mathfrak{R}^{01}_{\sigma s_1}(\lambda - z_1) \tag{12}$$

$$T(\lambda|z) = \mathrm{tr}_0 J(\lambda|z). \tag{13}$$

Here the trace is taken in the auxiliary space V_σ and is denoted as tr_0. It is convenient to rewrite $\mathfrak{R}^{12}_{\sigma s}(\lambda)$ and $J(\lambda|z)$ as matrices in the auxilary space V_σ:

$$\mathfrak{R}_{\sigma s}(\lambda) = \begin{pmatrix} 2I_0 + 2\frac{W_3 I_3}{W_0} & \frac{2I_-}{W_0} \\ \frac{2I_+}{W_0} & 2I_0 - 2\frac{W_3 I_3}{W_0} \end{pmatrix} \tag{14}$$

$$J(\lambda|z)=\begin{pmatrix} A(\lambda|z) & B(\lambda|z) \\ C(\lambda|z) & D(\lambda|z)\end{pmatrix}. \tag{15}$$

In (14) we redefined the $\Re_{\sigma_i}(\lambda)$, dividing by $W_0(\lambda)/2$. From equation (6) we then have

$$\Re^{12}(\lambda-\mu)(J_1(\lambda|z)\otimes J_2(\mu|z))=(J_2(\mu|z)\otimes J_1(\lambda|z))\Re^{12}(\lambda-\mu) \tag{16}$$

and also

$$[T(\lambda|z),T(\mu|z)]=0. \tag{17}$$

In (15) we introduced $\Re^{12}(\lambda)=P^{12}\Re^{12}_{\sigma}(\lambda)$, where P^{12} is the permutation operator. Rewriting (16) in components, we get the commutation relations between the elements $J(\lambda|z)$. We need the following:

$$\begin{aligned}&[A(\lambda|z),A(\mu|z)]=[D(\lambda|z),D(\mu|z)]=0\\&[B(\lambda|z),B(\mu|z)]=[C(\lambda|z),C(\mu|z)]=0\end{aligned} \tag{18}$$

$$A(\mu|z)B(\lambda|z)=\frac{1}{c(\lambda-\mu)}B(\lambda|z)A(\mu|z)-\frac{b(\lambda-\mu)}{c(\lambda-\mu)}B(\mu|z)A(\lambda|z) \tag{19}$$

$$D(\mu|z)B(\lambda|z)=\frac{1}{c(\mu-\lambda)}B(\lambda|z)D(\mu|z)-\frac{b(\mu-\lambda)}{c(\mu-\lambda)}B(\mu|z)D(\lambda|z) \tag{20}$$

where

$$b(\lambda)=\frac{\sin(2\eta)}{\sin(\lambda+2\eta)}\qquad c(\lambda)=\frac{\sin(\lambda)}{\sin(\lambda+2\eta)}. \tag{21}$$

Using these commutation relations we find the off-shell Bethe ansatz equation (OSBAE):

$$T(\lambda|z)\Phi(\lambda_1,\ldots,\lambda_m|z)=\Lambda(\lambda,\lambda_1,\ldots,\lambda_m|z)\Phi(\lambda_1,\ldots,\lambda_m|z)-\sum_{\alpha=1}^{m}\frac{F_\alpha\Phi_\alpha}{\sin(\lambda-\lambda_\alpha)}. \tag{22}$$

Here $\Phi(\lambda_1,\ldots,\lambda_m|z)$ is the Bethe wave vector

$$\Phi(\lambda_1,\ldots,\lambda_m|z)=\prod_{\alpha=1}^{m}B(\lambda_\alpha|z)|\Omega\rangle \tag{23}$$

$$|\Omega\rangle=\prod_{i=1}^{N}\otimes|s_i,s_i\rangle\qquad S_i^3|s_i,s_i\rangle=s_i|s_i,s_i\rangle \tag{24}$$

and $\Phi_\alpha=\Phi(\lambda_1,\ldots,\lambda_{\alpha-1},\lambda,\lambda_{\alpha+1},\ldots,\lambda_m|z)$. The functions Λ and F_α are defined as follows:

$$\begin{aligned}\Lambda(\lambda,\lambda_1,\ldots,\lambda_m|z)=&\prod_{i=1}^{N}\frac{\sin(\lambda-z_i+\eta(1+2s_i))}{\cos(\eta)\sin(\lambda-z_i+\eta)}\prod_{\alpha=1}^{m}\frac{\sin(\lambda_\alpha-\lambda+2\eta)}{\sin(\lambda_\alpha-\lambda)}\\&+\prod_{i=1}^{N}\frac{\sin(\lambda-z_i+\eta(1-2s_i))}{\cos(\eta)\sin(\lambda-z_i+\eta)}\prod_{\alpha=1}^{m}\frac{\sin(\lambda-\lambda_\alpha+2\eta)}{\sin(\lambda-\lambda_\alpha)}\end{aligned} \tag{25}$$

$$\begin{aligned}F_\alpha(\lambda_1,\ldots,\lambda_m|z)=&\sin(2\eta)\prod_{i=1}^{N}\frac{\sin(\lambda_\alpha-z_i+\eta(1+2s))}{\cos(\eta)\sin(\lambda_\alpha-z_i+\eta)}\prod_{\beta\neq\alpha}^{m}\frac{\sin(\lambda_\alpha-\lambda_\beta-2\eta)}{\sin(\lambda_\alpha-\lambda_\beta)}\\&-\sin(2\eta)\prod_{i=1}^{N}\frac{\sin(\lambda_\alpha-z_i+\eta(1+2s))}{\cos(\eta)\sin(\lambda_\alpha-z_i+\eta)}\prod_{\beta\neq\alpha}^{m}\frac{\sin(\lambda_\alpha-\lambda_\beta+2\eta)}{\sin(\lambda_\alpha-\lambda_\beta)}.\end{aligned} \tag{26}$$

The next step in the Bethe ansatz would consist of imposing the vanishing of the 'unwanted' term in (22) such that $F_\alpha=0$. The Bethe ansatz equations $F_\alpha=0$ classify the eigenvalues and eigenvectors of the $T(\lambda|z)$. One can say that when the Bethe ansatz equations $F_\alpha = 0$ are satisfied, the wavefunction is on mass shell. If the conditions $F_\alpha = 0$ are not imposed, then, in general, we have equation (22), and the Bethe wavefunction is off mass shell. In this case, as in the rational situation [8–9], we call equation (22) the off-shell Bethe ansatz equation (OSBAE).

3. OSBAE and Gaudin magnets

By quasiclassical expansion one commonly performs the expansion of the vertex weight $\Re(\lambda, \eta)$ around some point η_0, such that $\Re^{12}(\lambda, \eta_0) = I^1 \otimes I^2$ [20]. One can parametrize η, such that $\eta_0 = 0$. It is easy to calculate the quasiclassical expansion of $\Re_{\sigma s}(\lambda)$. From (11)–(12) we have

$$\Re^{12}_{\sigma s}(\lambda) = I^1 \otimes I^2 + \eta r^{12}(\lambda) + \mathrm{O}(\eta^2) \tag{27}$$

$$r^{12}(\lambda) = \frac{2}{\sin(\lambda)}\left[\sigma_1^1 \otimes S_1^2 + \sigma_1^2 \otimes S_2^2 + \cos(\lambda)\sigma_1^3 \otimes S_2^3\right]. \tag{28}$$

The $r^{12}(\lambda)$ is the classical r-matrix and satisfies the classical Yang–Baxter equation

$$[r^{12}(\lambda - \mu), r^{13}(\lambda) + r^{23}(\mu)] + [r^{13}(\lambda), r^{23}(\mu)] = 0. \tag{29}$$

Let us introduce $J_k(\lambda) = J(\lambda = z_k|z), T_k = T(\lambda = z_K|z)$ and $\Lambda_k = \Lambda(\lambda = z_k, \lambda_1, \ldots, \lambda_m|z)$. From (12)–(15) and using (27)–(28) we find the quasiclassical expansion of all the objects in the OSBAE:

$$T_k = 2 + 8\eta \sum_{j\neq k}^{N} \frac{1}{\sin(z_k - z_j)}\left[S_k^1 \otimes S_j^1 + S_k^2 \otimes S_j^2 + \cos(z_k - z_j) S_k^3 \otimes S_j^3\right] + \mathrm{O}(\eta^2) \tag{30}$$

$$\lambda_k = 2 + 8\eta \sum_{i\neq k}^{N} s_k s_i \cot(z_k - z_i) + 8\eta \sum_{\alpha=1}^{m} s_k \cot(\lambda_\alpha - z_k) + \mathrm{O}(\eta^2) \tag{31}$$

$$F_\alpha = 8\eta^2 \sum_{\beta\neq\alpha}^{m} \cot(\lambda_\alpha - \lambda_\beta) - 8\eta^2 \sum_{i=1}^{N} s_i \cot(\lambda_\alpha - z_i) + \mathrm{O}(\eta^3) \tag{32}$$

$$B(\lambda|z) = 2\eta \sum_{i=1}^{N} \frac{S_i^-}{\sin(\lambda - z_i)} + \mathrm{O}(\eta^2) = 2\eta S^-(\lambda) + \mathrm{O}(\eta^2) \tag{33}$$

$$B(\lambda = z_k) = 2S_k^- + \mathrm{O}(\eta) \tag{34}$$

$$\Phi(\lambda_1, \ldots, \lambda_m|z) = (2\eta)^m \prod_{\alpha=1}^{m} S^-(\lambda_\alpha)|\Omega\rangle + \mathrm{O}(\eta^{m+1}) \tag{35}$$

$$\Phi_\alpha = (2\eta)^{m-1} 2S_k^- \prod_{\beta\neq\alpha}^{m} S^-(\lambda_\beta) + \mathrm{O}(\eta^m). \tag{36}$$

After setting $\lambda = z_k$ in (22), then using (31)–(37) and combining the terms proportional to η^{m+1} we find the quasiclassical limit (the first non-trivial consequence) of the OSBAE (22):

$$H_k\varphi = h_k\varphi - \sum_{\alpha=1}^{m} \frac{f_\alpha S_k^- \varphi'_\alpha}{\sin(z_k - \lambda_\alpha)}. \tag{37}$$

Here the operators H_k are the Gaudin [2, 3] non-local magnets:

$$H_k = \sum_{i \neq k}^{N} \frac{1}{\sin(z_k - z_i)} \left[S_k^1 \otimes S_i^1 + S_i^k \otimes S_i^2 + \cos(z_k - z_t) S_k^3 \otimes S_i^3 \right]. \tag{38}$$

In (37) we introduced the new notations

$$h_k = \sum_{i \neq k}^{N} s_k s_i \cot(z_k - z_i) + \sum_{\alpha=1}^{m} s_k \cot(\lambda_\alpha - z_k) \tag{39}$$

$$f_\alpha = \sum_{\beta \neq \alpha}^{m} \cot(\lambda_\alpha - \lambda_\beta) - \sum_{i=1}^{N} s_i \cot(\lambda_\alpha - z_i) \tag{40}$$

$$|\varphi\rangle = \prod_{\alpha=1}^{m} S^-(\lambda_\alpha)|\Omega\rangle. \tag{41}$$

In (37) we also introduced $\varphi = S^-(\lambda_\alpha)\varphi'_\alpha$. The equations (37)–(41) reproduce the Gaudin [2, 3] results, which he found considering the spectral problem for the set of non-local commuting Hamiltonians H_k. If in (37) we impose the condition $f_\alpha = 0$, then we obtain φ as eigenvector of the operators H_k with eigenvalues h_k. The parameters $\lambda_1, \ldots, \lambda_m$ have to be found from the quasiclassical Bethe ansatz equations $f_\alpha = 0$.

4. Trigonometric Knizhnik–Zamolodchikov equation

Let us introduce the function $\chi(\lambda|z) = \chi(\lambda_1, \ldots, \lambda_m|z_1, \ldots, z_N)$, as in the rational case [7–9], obeying the following differential equations

$$\kappa \frac{\partial \chi}{\partial z_j} = h_j \chi \tag{42}$$

$$\kappa \frac{\partial \chi}{\partial \lambda_\alpha} = f_\alpha \chi \tag{43}$$

where κ is some constant. The zero-curvature conditions are fulfilled:

$$\frac{\partial h_j}{\partial \lambda_\alpha} = \frac{\partial f_\alpha}{\partial z_j}. \tag{44}$$

The solution of the equations (42), (43) is given by

$$\chi(\lambda|z) = \prod_{i \neq j}^{N} (\sin(z_i - z_j))^{s_i s_j/\kappa} \prod_{\alpha \neq \beta}^{m} (\sin(\lambda_\alpha - \lambda_\beta))^{1/\kappa} \prod_{k\gamma} (\sin(z_k - \lambda_\gamma))^{-s_k/\kappa}. \tag{45}$$

As in the rational case, we define the vector-function $\Psi(z_1, \ldots, z_N)$ through multiple contour integrals as follows:

$$\Psi(z_1, \ldots, z_N) = \oint \ldots \oint \chi(\lambda|z)\varphi(\lambda|z) \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_m. \tag{46}$$

Here the integrations are to be taken over some cycles. Now it is easy to show that the vector-function (46) is a solution of the trigonometric Knizhnik–Zamolodchikov equation [15]:

$$\kappa \frac{\partial \Psi}{\partial z_k} = H_k \Psi. \tag{47}$$

Here H_k are the Gaudin trigonometric Hamiltonians (38). Substituting (46) into the Knizhnik–Zamolodchikov equation (47), using OSBAE (37) and (42), (43) and finally by taking into account the relation

$$\frac{\partial \varphi}{\partial z_k} = -\sum_{\alpha=1}^{m} \frac{\partial}{\partial \lambda_\alpha} \frac{S_k^- \varphi'_\alpha}{\sin(\lambda_\alpha - z_k)} \tag{48}$$

we find the equation

$$\sum_{\alpha=1}^{m} \oint \dots \oint \frac{\partial}{\partial \lambda_\alpha} \left[\frac{\chi S_k^- \varphi'_\alpha}{\sin(z_k - \lambda_\alpha)} \right] \mathrm{d}\lambda_1 \dots \lambda_m = 0. \tag{49}$$

It is evident that this equation is satisfied because the contours are closed.

5. Quantum Knizhnik–Zamolodchikov equation

The quantum Knizhnik–Zamolodchikov equation was introduced in [13] as a difference equation for the correlation function of the interwining operators and in [12] also as a difference equation for the form factor in integrable massive field theory. In formal language it is a difference equation for the vector-valued function $f(z_1, \dots, z_N) \in V^{l_1}, \dots, V^{l_N}$, where V^{l_i} is a representation space of the $U_q(\mathfrak{G})$. In this section we analyse the connection between the solution of the quantum Knizhnik–Zamolodchikov equation corresponding to $U_q(\widehat{sl(2)})$ with our solution of the trigonometric Knizhnik–Zamolodchikov equation. The quantum Knizhnik–Zamolodcikov equation is the system of the following difference equations [13, 14]

$$f(z_1, \dots, z_j + \kappa, \dots, z_N) = A_j(z_1, \dots, z_N) f(z_1, \dots, z_N) \tag{50}$$

where the operators A_j acting on $V^{l_1}, \dots, V^{l_N}$ are given by

$$\begin{aligned} A_j(z_1, \dots, z_N) = {} & \mathfrak{R}^{l_j l_{j-1}}(z_j - z_{j-1} + \kappa), \dots, \mathfrak{R}^{l_j l_1}(z_j - z_1 + \kappa), Z^{(l_j)}, \\ & \mathfrak{R}^{l_j l_N}(z_j - z_N), \dots, \mathfrak{R}^{l_j l_{j+1}}(z_j - z_{j+1}). \end{aligned} \tag{51}$$

In (50), (51) κ is some constant, $\mathfrak{R}^{l_i l_j}(\lambda)$ satisfies the Yang–Baxter equation (8), but the spaces 1, 2, 3 in general are different. The matrices $Z^{(l_j)}$ are defined as follows:

$$[Z^l \otimes Z^m, \mathfrak{R}^{lm}(\lambda)] = 0. \tag{52}$$

In [14] the solution of the quantum Knizhnik–Zamolodchikov (50) was written as the following Jackson-type integral:

$$f(z_1, \dots, z_N) = \sum_{(\lambda_1, \dots, \lambda_m)} g(\lambda|z) \Phi(\lambda_1, \dots, \lambda_m|z) \tag{53}$$

where $\Phi(\lambda_1, \dots, \lambda_m|z)$ is the Bethe wavevector (23) and the function $g(\lambda|z)$ is given as

$$g(\lambda|z) = \prod_{i \neq j}^{N} P^{l_i, l_j}(z_i - z_j) \prod_{i=1}^{N} \prod_{\alpha=1}^{m} D^{l_i}(\lambda_\alpha - z_i) \prod_{\alpha \neq \beta}^{m} Q(\lambda_\alpha - \lambda_\beta) \exp\left(c \sum_{\alpha=1}^{m} \lambda_\alpha \right). \tag{54}$$

When all $l_i = 1, i = 1, \dots, N$, such that the case that all the spins are half in our language, the functions $P^{l_i l_j}(x)$, $D^{l_j}(x)$ and $Q(x)$ corresponding to $U_q(\widehat{sl(2)})$ satisfiy the following difference equations [14]:

$$D(x + \kappa) = \frac{\exp(\mathrm{i}x) - q^2}{\exp(\mathrm{i}x) - 1} q^{-1} D(x) \tag{55}$$

$$Q(x+\kappa) = \frac{q\exp(\mathrm{i}x) - q^{-1}}{q^{-1}\exp(\mathrm{i}x+\kappa) - q}\,\frac{\exp(\mathrm{i}x+\kappa) - 1}{\exp(\mathrm{i}x) - 1}\,Q(x) \tag{56}$$

$$P(x+\kappa) = \frac{q\exp(\mathrm{i}x) - q^{-1}}{\exp(\mathrm{i}x) - 1}\,P(x). \tag{57}$$

If we put $\kappa = \eta\kappa$ and $q = \exp(-\mathrm{i}\eta/2)$ in (55)–(57), and after we take the limit $\eta \to 0$, we obtain the differential equations

$$\kappa\frac{\mathrm{d}D}{\mathrm{d}x} = \frac{1}{2}\frac{\cos(x/2)}{\sin(x/2)}D(x) \tag{58}$$

$$\kappa\frac{\mathrm{d}Q}{\mathrm{d}x} = -\frac{\cos(x/2)}{\sin(x/2)}Q(x) \tag{59}$$

$$\kappa\frac{\mathrm{d}P}{\mathrm{d}x} = -\frac{1}{2}\frac{\cos(x/2)}{\sin(x/2)}P(x). \tag{60}$$

The solutions are given as

$$\begin{aligned} D(x) &= (\sin(x/2))^{-1/\kappa} \\ Q(x) &= (\sin(x/2))^{2/\kappa} \\ P(x) &= (\sin(x/2))^{1/\kappa}. \end{aligned} \tag{61}$$

In equation (54) it is necessary to make the choice $c = c'\eta$ and then to take the limit $\eta \to 0$. So from (54) and (61) we conclude that in the quasiclassical limit we have

$$g(\lambda|z) \to \chi(\lambda|z) \tag{62}$$

where $\chi(\lambda|z)$ is given by (45). Taking into account that the Bethe wavevector $\Phi(\lambda_1,\ldots,\lambda_m|z)$ in the quasiclassical limit is of the form (35), we conclude that in the quasiclassical limit the solution of the quantum Knizhnik–Zamolodchikov equation is equal to the solution of the trigonometric Knizhnik–Zamolodchikov equation (46) in the sense that

$$f(z_1,\ldots,z_N) \to (2\eta)^m\Psi(z_1,\ldots,z_N). \tag{63}$$

In general, the quasiclassical limit of the solution (53) of the quantum Knizhnik–Zamolodchikov equation is connected with the solution of the trigonometric Knizhnik–Zamolodchikov equation as given by Cherednik [15]

$$\kappa\frac{\partial\Psi}{\partial z_k} = \sum_{i\neq k}^{N} r_{ki}(z_k - z_i)\Psi \tag{64}$$

where $r_{ki}(\lambda)$ is a classical r-matrix. In order to find a solution of equation (64) in our language it is necessary to use the OSBAE for the $T_s(\lambda)$ constructed from the solution $\mathfrak{R}_s(\lambda)$ (8). Note that the $T_s(\lambda)$ obey the equation $[T(\lambda), T_s(\mu)] = 0$ and that they have the same Bethe wavevector $\Phi(\lambda_1,\ldots,\lambda_m|z_1,\ldots,z_N)$.

6. Conclusion

In this paper we have constructed a solution of the trigonometric Kniznik–Zamolodchikov equation with use of the OSBAE for the inhomogeneous *XXZ* vertex model. The solution obtained here is the quasiclassical limit of the solution to the quantum Knizhnik–Zamolodchikov equation corresponding to $U_q(\widehat{sl}_2)$. From this observation we conclude that all Bethe wavevectors, which satisfy the OSBAE, generate the solutions of the quantum

Knizhnik–Zamolodchikov equation. In the quasiclassical limit, when the Bethe wavevectors become equal to the Bethe wavevectors for the Gaudin magnets, they generate the solutions of the usual (undeformed) Knizhnik–Zamolodchikov equation (rational, trigonometric, and elliptic). It is intriguing to understand the physical meaning of this fine structure. Why do Bethe wavevectors for the inhomogeneous vertex model lead to the solution of the Knizhnik–Zamolodchikov equations (usual and deformed)? (During the preparation of this paper we learned that a similar result (when all the spins $s_i = \frac{1}{2}, i = 1, \ldots, N$) was obtained in [21].)

Acknowledgments

The author would like to thank Professor Schrader for his hospitality during his stay at the Institut für Theoretische Physik (FUB). This visit was financed by the VW Project 'Cooperation with scientists from CIS'. Also he would like to thank M Karowski, M Schmidt and R Schrader for stimulating discussions.

References

[1] Baxter R J 1982 *Exactly solved models in statistical mechanics* (London: Academic)
[2] Gaudin M 1976 *J. Physique (Paris)* **37** 1087–98
[3] Gaudin M 1983 *Le function d'Onde de Bethe* (Paris: Masson)
[4] de Vega H J 1989 *Int. J. Mod. Phys.* A **4** 2371
[5] Luther A 1976 *Phys. Rev.* B **14** 2153
Luscher M 1976 *Phys. Rev.* B **117** 475
[6] Faddeev L D and Takhtajan L A 1979 *Usp. Math. Nauk* **34** 5–63 (in Russian)
[7] Babujian H M 1990 Correlation function in WZNW model as a Bethe wave function for the Gaudin magnetics *Proc. XXIV Int. Symp. (Ahreschoop, Zeuten, 1990)* (*Preprint* YERPHI-1261) pp 47–90
[8] Babujian H M 1993 *J. Phys. A: Math. Gen.* **26** 3–71
[9] Babujian H M and Flume R 1993 Off-shell Bethe ansatz equation for Gaudin magnets and solution of Knizhnik–Zamolodchikov equations *Preprint* Bonn-HE-93-90
[10] Knizhnik V and Zamolodchikov A 1984 *Nucl. Phys.* B **247** 83–103
[11] Schechtman V and Varchenko A 1991 *Invent. Math.* **106** 139–94
[12] Smirnov F A 1991 *Form factors in completely integrable models of quantum field theory* (Singapore: World Scientific)
[13] Frenkel I B and Reshetikhin N Yu 1992 *Comm. Math. Phys.* **146** 1–60
[14] Reshetikhin N Yu 1992 *Lett. Math. Phys.* **26** 153–65
[15] Cherednik I 1991 *Publ. Rims. Kyoto Univ.* **27** 727–44
[16] Babujian H M and Tsvelick A M 1986 *Nucl. Phys.* B **265** 24–44
[17] Kirilov A N and Reshetikin N Yu 1987 *J. Phys. A: Math. Gen.* **20** 1565–85
[18] Sklyanin E K 1983 *Funktional. Anal. Prilozhin.* **17** 34–48
[19] Jimbo M 1985 *Lett. Math. Phys.* **10** 63–91
[20] Belavin A A and Drinfeld V G 1982 *Funct. Anal. Appl.* **16** 159
[21] Hasegawa T, Hikami K and Wadati M 1994 Integral solution of trigonometric Knizhnik–Zamolodchikov equation; off-shell Bethe ansatz method *Preprint* Tokyo University